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Exact Solution of a Case of Semiclassical Laser Equations

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This paper solves a special case of the equations for Lamb's semiclassical laser theory. The results are used to discuss the asymptotic behavior of the solution for large electromagnetic field and the power-series expansion in the field amplitude of the population inversion. A comparison with the continued-fraction method used earlier is made.

I. INTRODUCTION

A RECENT paper[1] gives a solution to the equations of the semiclassical Lamb model of a laser[2] for the case where only one cavity mode is oscillating. The solution emerges in the form of a continued fraction which can be used for computations even at very high intensities of the electromagnetic field in the mode at which the laser oscillates. The Lamb model has been very useful in the treatment of various problems in the theory of the laser (for a list of references and a discussion of some recent experiments, see Ref. 1). It is hence interesting to find that a special case of the laser equations can be solved in analytical form. This is the case when the atomic transitions are in exact resonance with the electromagnetic mode of the laser cavity, and the two atomic levels involved in the transition decay with exactly the same decay constants to lower levels. The first condition can be realized in every laser by tuning the cavity, whereas the second is an improbable coincidence, which, however, cannot be expected to lead to any special physical effects distinguishing it from the case of nearly the same decay constants for the two levels.

The solution is given in terms of an integral which has not been evaluated analytically. Consequently, numerical results are more easily obtainable from the results of Ref. 1, but the analytical solution is useful in answering questions of a general nature. In this paper the laser equations of the semiclassical model are presented in Sec. II; in Sec. III, they are solved for the special case we are considering. In Sec. IV, the solution is used to obtain the asymptotic behavior of the laser for very large intensities, a question that remained unsettled in Ref. 1. In Sec. V, the population inversion density is expanded in a power series in the electromagnetic field, and the convergence properties are discussed. Finally a short discussion is presented in Sec. VI.

II. BASIC LASER EQUATIONS

In Ref. 1 it is shown that the steady-state amplitude of the electromagnetic field in the laser is given by the equation

$$E=-(Q/\epsilon_0)S\,, \tag{1}$$

where Q is the laser cavity Q value and ϵ_0 is the dielectric constant. The quantity S is given by the velocity average and mode projection of a function $S(z,v,t,\bar{t})$:

$$S=\int_{-\infty}^{+\infty} dv\, W(v)(2/L)\int_0^L dz\,\sin Kz\, S(z,v,t,\bar{t})\,|_{\bar{t}=t}\,, \tag{2}$$

where L is the length of the laser (taken to be along the z axis). S depends also on the field F so that Eq. (1) determines E only in an implicit way.

The function $S(z,v,t,\bar{t})$ is determined from the coupled integrodifferential equations

$$\begin{aligned}(\partial/\partial\bar{t})S(z,v,t,\bar{t})&=-\tfrac{1}{2}(\gamma_a+\gamma_b)S(z,v,t,\bar{t})-(\omega-\nu)^2\\&\quad\times\int_0^\infty S(z,\,v,\,t,\,\bar{t}-\tau)\,\exp[-\tfrac{1}{2}(\gamma_a+\gamma_b)\tau]d\tau\\&\quad-(\wp^2E/\hbar)\sin\{K[z-v(t-\bar{t})]\}\,N(z,v,t,\bar{t})\,,\end{aligned} \tag{3}$$

$$\begin{aligned}(\partial/\partial\bar{t})N(z,v,t,\bar{t})&=-\tfrac{1}{2}(\gamma_a+\gamma_b)N(z,v,t,\bar{t})+\tfrac{1}{4}(\gamma_a-\gamma_b)^2\\&\quad\times\int_0^\infty N(z,\,v,\,t,\,\bar{t}-\tau)\,\exp[-\tfrac{1}{2}(\gamma_a+\gamma_b)\tau]d\tau\\&\quad+(E/\hbar)\sin\{K[z-v(t-\bar{t})]\}\,S(z,v,t,\bar{t})\\&\quad+(\gamma_b\Lambda_a-\gamma_a\Lambda_b)/[\tfrac{1}{2}(\gamma_a+\gamma_b)]\,,\end{aligned} \tag{4}$$

where Λ_a and Λ_b are the pumping rates to the upper and lower atomic levels a and b; and γ_a, γ_b are their decay rates. The energy $\hbar\omega$ is the difference between the energies of the two levels and $\wp$ is the dipole matrix element coupling them. The frequency ν is the oscillation frequency of the laser, and it is very close to the resonance frequency Ω of the cavity. Equations (3) and (4) are derived from the equation of the motion for the density matrix of a two level system and the quantity $S(z,v,t,\bar{t})$ gives the polarization of the atomic system under the influence of the electromagnetic field. The function $N(z,v,t,\bar{t})$, on the other hand, can be used to obtain the distribution of the difference in population between the two levels (the population inversion

[1] S. Stenholm and W. E. Lamb, Jr., Phys. Rev. **181**, 618 (1969).
[2] W. E. Lamb, Jr., Phys. Rev. **134**, A1429 (1964).

density). Taking the velocity average

$$N(z)=\int_{-\infty}^{+\infty} dv\, W(v)N(z,v,t,\bar{t})\,|_{\bar{t}=t}\,, \qquad (5)$$

we obtain the spatial distribution of the inversion density, and taking the spatial average

$$N(v)=(1/L)\int_0^L dz\, N(z,v,t,\bar{t})\,|_{\bar{t}=t}\,, \qquad (6)$$

we obtain the velocity distribution. The function $N(z,v,t,\bar{t})$ contains the features described as "holes in the population" burned by the electromagnetic field. The results presented above are derived and discussed in Ref. 1 and the basis for the semiclassical model is presented by Lamb in Ref. 2.

By tuning the optical cavity we can achieve exact resonance $\Omega=\nu=\omega$. If we further assume that $\gamma_a=\gamma_b\equiv\gamma$, the integral terms disappear from Eqs. (3) and (4) and we are left with two coupled differential equations. These are considerably easier to treat, and an analytical solution is possible. The case of equal atomic decay constants can hardly be found in nature. The unequal decay rates make it possible to achieve a population inversion even with equal pumping rates to the two levels, $\Lambda_a=\Lambda_b\equiv\Lambda$, because then the inversion at zero electromagnetic field,

$$\bar{N}=\Lambda_a/\gamma_a-\Lambda_b/\gamma_b=\Lambda(\gamma_a^{-1}-\gamma_b^{-1})\,, \qquad (7)$$

is positive as soon as the lower level decays more rapidly than the upper, i.e., $\gamma_b>\gamma_a$. When this effect is taken into account by the introduction of effective pumping rates Λ_a, Λ_b such that they give the right value for $\bar{N}$, it is expected that a small difference between γ_a and γ_b can be neglected. In real lasers they are always of the same order of magnitude.

III. SOLUTION OF EQUATIONS

With $\omega=\nu$ and $\gamma_a=\gamma_b$, Eqs. (3) and (4) [with the arguments $(z,v,t,\bar{t})$ of S and N omitted] are

$$(\partial S/\partial\bar{t})=-\gamma S-(\wp^2E/\hbar)\sin\{K[z-v(t-\bar{t})]\}N\,, \qquad (8)$$

$$(\partial N/\partial\bar{t})=-\gamma N+(E/\hbar)\sin\{K[z-v(t-\bar{t})]\}S+\gamma\bar{N}\,, \qquad (9)$$

with the adjusted value of $\bar{N}$.

To write the equations in a more symmetric form, we introduce the variables a and b by setting

$$S=\wp\bar{N}a\,,\quad N=\bar{N}b\,, \qquad (10)$$

and use the abbreviation

$$F(\bar{t})=(\wp E/\hbar)\sin\{K[z-v(t-\bar{t})]\}\,. \qquad (11)$$

The resulting equations can be written in matrix form

$$\frac{\partial}{\partial\bar{t}}\begin{bmatrix}a\\b\end{bmatrix}=-\gamma\begin{bmatrix}a\\b\end{bmatrix}+F(\bar{t})\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}+\gamma\begin{bmatrix}0\\1\end{bmatrix}. \qquad (12)$$

The solution of this equation which is zero when $\bar{t}=-\infty$ is given by

$$\begin{bmatrix}a\\b\end{bmatrix}=\gamma\int_{-\infty}^{\bar{t}}\exp\left\{\gamma(t'-\bar{t})+\int_{t'}^{\bar{t}}F(\tau)d\tau\begin{bmatrix}0&-1\\1&0\end{bmatrix}\right\}\begin{bmatrix}0\\1\end{bmatrix}dt'$$

$$=\gamma\int_0^\infty e^{-\gamma\eta}\exp\left\{\int_{\bar{t}-\eta}^{\bar{t}}F(\tau)d\tau\begin{bmatrix}0&-1\\1&0\end{bmatrix}\right\}\begin{bmatrix}0\\1\end{bmatrix}d\eta. \qquad (13)$$

It is possible to evaluate the function

$$\theta(\eta)=\int_{\bar{t}-\eta}^{\bar{t}}F(\tau)d\tau=\int_{\bar{t}-\eta}^{\bar{t}}\frac{\wp E}{\hbar}\sin K[z-v(t-\tau)]d\tau$$

$$=(\wp E/\hbar Kv)\{\cos K[z-v(t-\bar{t}+\eta)]$$

$$-\cos K[z-v(t-\bar{t})]\}\,, \qquad (14)$$

and then the solution is formally given by (13). In the physically meaningful relations (1), (5), and (6), we have to set $\bar{t}=t$ and this may now be done in (14), giving

$$\theta(\eta)=(\wp E/\hbar Kv)[\cos(Kz-Kv\eta)-\cos Kz]$$

$$=(\wp E/\hbar Kv)[\cos Kz(\cos Kv\eta-1)$$

$$+\sin Kz\cos Kv]. \qquad (15)$$

As the square of the matrix in the exponential of Eq. (13) is minus the unit matrix

$$\begin{bmatrix}0&-1\\1&0\end{bmatrix}^2=-\begin{bmatrix}1&0\\0&1\end{bmatrix}, \qquad (16)$$

it is possible to rearrange the terms in the series expansion of the potential to give the sine and cosine series, and, consequently,

$$\exp\left\{\theta(\eta)\begin{bmatrix}0&-1\\1&0\end{bmatrix}\right\}=\cos\theta(\eta)+\sin\theta(\eta)\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

$$=\begin{bmatrix}\cos\theta(\eta)&-\sin\theta(\eta)\\\sin\theta(\eta)&\cos\theta(\eta)\end{bmatrix}. \qquad (17)$$

Inserting (17) into (13) and performing the matrix multiplication, we obtain

$$\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}-\gamma\int_0^\infty e^{-\gamma\eta}\sin\theta(\eta)d\eta\\ \gamma\int_0^\infty e^{-\gamma\eta}\cos\theta(\eta)d\eta\end{bmatrix}. \qquad (18)$$

Equations (18) give the solution to the special case under consideration, but because of the complicated form of $\theta(\eta)$ in (15), the expressions in (18) are not simple to evaluate. The intensity E is determined implicitly by the equation following from (1), (10), (18), and (15):

$$E=\frac{\wp Q\gamma\bar{N}}{\epsilon_0}\int_{-\infty}^{\infty}dv\,W(v)\frac{2}{L}\int_0^L dz\,\sin Kz\int_0^\infty d\eta\,e^{-\gamma\eta}\times\sin\left(\frac{\wp E}{\hbar Kv}[\cos(Kz-Kv\eta)-\cos Kz]\right),\quad(19)$$

and similarily (18) and (10) inserted into (6) give the atomic population inversion

$$N(v)=\frac{\bar{N}\gamma}{L}\int_0^L dz\int_0^\infty d\eta\,e^{-\gamma\eta}\times\cos\left\{\frac{\wp E}{\hbar}\left[\frac{\cos Kv\eta-1}{Kv}\cos Kz+\frac{\sin Kv\eta}{Kv}\sin Kz\right]\right\}.\quad(20)$$

A more detailed discussion of the structure of Eq. (19) is deferred to Sec. VI. First we are going to use the analytic solution (18) to discuss two questions which were considered in Ref. 1 but which remained unsettled there, namely, the asymptotic behavior of (19) for large electromagnetic fields and a series expansion of Eq. (20).

IV. ASYMPTOTIC BEHAVIOR FOR LARGE FIELDS

In Ref. 1 it is suggested that for large intensities of the electromagnetic field we have the proportionality $E^2\propto\bar{N}$. This statement could not be proved there, and the numerical results were inconclusive because they were obtained in the Doppler limit when the Lorentzian linewidth γ of the atomic transitions is much smaller than the width of the velocity distribution $W(v)$. In this case it is known (see Ref. 1) that the asymptotic behavior $E^2\propto\bar{N}^2$ is valid over a large range of values for the intensity. It is thus of interest to see what the asymptotic behavior of the integral in Eq. (19) is.

The expressions in Eq. (18) contain the integrals of exponentials which can be written

$$\gamma\int_0^\infty e^{-\gamma\eta}\exp\left\{i\frac{\wp E}{\hbar Kv}[\cos(Kz-Kv\eta)-\cos Kz]\right\}d\eta=s\int_0^\infty e^{-sx}\exp\{iT[\cos(y-x)-\cos y]\}dx,\quad(21)$$

with the notation

$$y=Kz,\quad x=Kv\eta,\quad s=\gamma/Kv,\quad T=\wp E/\hbar Kv.\quad(22)$$

For a fixed value of v, the large field limit implies $T\to\infty$, and the asymptotic value of the integral

$$I_1(y)=\int_0^\infty e^{-sx}\exp\{iT[\cos(x-y)-\cos y]\}dx\quad(23)$$

has to be calculated. [The arguments s and T are not explicitly written out in $I_1(y)$.]

When T is large, the main contribution to the integral in (23) comes from the end points and the points of stationary phase.[3] The end point $x=\infty$ does not contribute because of the factor e^{-sx}. The points of stationary phase are given by the equation

$$(d/dx)\cos(x-y)=-\sin(x-y)=0,\quad(24)$$

and the solutions are

$$x=y+n\pi\quad(25)$$

for all integers n such that $y+n\pi>0$. Each point of stationary phase gives a contribution proportional to $T^{-1/2}$ but the subsequent mode projection in Eq. (19),

$$\int_0^{KL}\sin y\,I_1(y)dy\propto\int_0^{KL}\sin y\,e^{iT\cos y}dy,\quad(26)$$

is found to introduce an additional factor T^{-1}. The calculations are performed in some detail in the Appendix. The total contribution to (19) from the points of stationary phase goes like $T^{-3/2}$ and is smaller than the contribution from $x=0$ for large enough values of T. The reason for this is that, when $x\approx0$, the two terms multiplying T in the exponential of Eq. (21) cancel and there is no exponential dependence on T left to the y integration. Consequently, this integral does not change the asymptotic dependence on T. We now calculate the contribution from the point $x=0$. We expand

$$\cos(y-x)-\cos y=(\sin y)x+\cdots,\quad(27)$$

where terms of order x^2 are neglected. We have

$$I_1(y)=\int_0^\infty e^{-sx}\,e^{iTx\sin y}dx=(s-iT\sin y)^{-1}.\quad(28)$$

Using this result and the relation $K=(n\pi/L)$ with n an integer, we can evaluate the integral

$$\begin{aligned}I_2&=\frac{2}{KL}\int_0^{KL}dy\,\sin y\,s\int_0^\infty e^{-xs}\times\sin\{T[\cos(y-x)-\cos y]\}dx\\&=\frac{2s}{n\pi}\int_0^{n\pi}dy\,\sin y\,(2i)^{-1}[I_1(y)-(I_1(y))^*]dy\\&=\frac{2s}{\pi}\int_0^\pi T\sin^2y\,(s^2+T^2\sin^2y)^{-1}dy.\end{aligned}\quad(29)$$

As $s=(\gamma/Kv)\ll T=(\wp E/\hbar Kv)$ in the asymptotic limit $E\to\infty$, we can write

$$I_2\sim 2s/T=2\hbar\gamma/\wp E,\quad(30)$$

[3] E. T. Copson, *Asymptotic Expansions* (Cambridge University Press, Cambridge, England, 1965), pp. 29–34.

with

$$s/T = h\gamma/\wp E \ll 1, \tag{31}$$

which implies that the dimensionless intensity parameter

$$I = \tfrac{1}{2}(\wp E/h\gamma)^2 \tag{32}$$

is large. Inserting (30) into Eq. (19), we get

$$E = \frac{\wp Q\bar{N}}{\epsilon_0}\frac{2h\gamma}{\wp E}\int_{-\infty}^{+\infty} W(v)dv, \tag{33}$$

since the integral I_2 no longer depends on the velocity.[4] Since the velocity distribution is normalized we find by using the intensity parameter of Eq. (32) that

$$I = \wp^2 Q\bar{N}/\epsilon_0\gamma h, \tag{34}$$

which is a result suggested in Ref. 1, Sec. 12, but not proved there.

The relation (34) is identical to the asymptotic result $I \to \infty$ for nonmoving atoms (see Ref. 1, Appendix A). This fact may be explained qualitatively if we extend the concept of power broadening to the limit of large field intensities. Then the atomic linewidth appears broader than the velocity distribution, which samples the emission line. This is the limit opposite to the Doppler limit when the atomic line samples the velocity distribution. At resonance between the atomic transitions and the laser cavity, the details of the velocity distribution become unimportant when $I \to \infty$, because the atomic line shape changes only slowly over the whole velocity curve. This velocity independence is evident in Eq. (30). Then the velocity integral gives 1, thanks to the normalization, and the result is the same as if all atoms were clustered at zero velocity. This picture explains both the velocity independence of I_2 and the coincidence of (34) with the zero-velocity result. But it also suggests that for the proportionality $I \propto \bar{N}$ to be valid, we have to go to very large intensities, especially in the Doppler limit.

V. POWER SERIES IN FIELD INTENSITY

In Ref. 1 it was found that a power-series expansion of the atomic population inversion density $N(v)$ cannot converge at $v=0$ for intensities $I>0.5$. Numerically, it was found that when $Kv<\gamma$ the convergence became so poor that the power-series expansion was useless. It is of interest to use the analytic expressions of this paper to discuss the convergence of the power series.

From Eq. (20), we find

$$N(v) = \frac{\bar{N}\gamma}{L}\int_0^L dz \int_0^\infty d\eta e^{-\gamma\eta} \times \cos\left(\frac{\wp E}{2h}[\chi^* e^{iKz} + \chi e^{-iKz}]\right), \tag{35}$$

where

$$\chi = (e^{iKv\eta} - 1)/Kv. \tag{36}$$

Expanding the cosine function and performing the z integration, we find

$$\begin{aligned} L^{-1}\int_0^L dz \cos\left(\frac{\wp E}{2h}(\chi^* e^{iKz} + \chi e^{-iKz})\right) &= L^{-1}\int_0^L dz \sum_{\alpha=0}^{\infty}(-1)^\alpha[(2\alpha)!]^{-1}\left(\frac{\wp E}{2h}\right)^{2\alpha} \\ &\quad \times \sum_{\beta=0}^{2\alpha}(2\alpha)![(2\alpha-\beta)!\beta!]^{-1}(\chi^* e^{iKz})^\beta(\chi e^{-iKz})^{2\alpha-\beta} \\ &= \sum_{\alpha=0}^{\infty}(-1)^\alpha(\alpha!)^{-2}\left(\frac{\wp E}{2h}\right)^{2\alpha}|\chi|^{2\alpha}, \end{aligned} \tag{37}$$

because

$$L^{-1}\int_0^L dz\, e^{inKz} = \delta_{n0}. \tag{38}$$

From (36) it follows that

$$|\chi|^2 = 4\sin^2(\tfrac{1}{2}Kv\eta)(Kv)^{-2} \tag{39}$$

and the η integration has the form

$$\gamma\int_0^\infty e^{-\gamma\eta}(\sin\tfrac{1}{2}Kv\eta)^{2\alpha}d\eta = \int_0^\infty e^{-x}\sin^{2\alpha}(\tfrac{1}{2}\kappa x)dx, \tag{40}$$

where

$$\kappa = Kv/\gamma \tag{41}$$

has been introduced. The integral (40) can be performed[5] to give

$$\int_0^\infty e^{-x}\sin^{2\alpha}(\tfrac{1}{2}\kappa x)dx = (2\alpha)!(\tfrac{1}{2}\kappa)^{2\alpha}(1+\kappa^2)^{-1} \times[1+(2\kappa)^2]^{-1}\cdots[1+(\alpha\kappa)^2]^{-1}, \tag{42}$$

which, introduced together with (37) into (35), gives

$$N(v) = \bar{N}\sum_{\alpha=0}^{\infty}(-1)^\alpha(2\alpha)!(\alpha!)^{-2} \times(\tfrac{1}{2}I)^\alpha(1+\kappa^2)^{-1}\cdots[1+(\alpha\kappa)^2]^{-1}, \tag{43}$$

when I has been introduced.

Setting $v=0$ is equal to setting $\kappa=0$ and gives the series expansion for

$$N(0) = \bar{N}(1+2I)^{-1/2}, \tag{44}$$

which is the correct result from Ref. 1, Appendix A. The expansion of (44) when $I>0.5$ is divergent, but as soon as $\kappa\neq 0$, we find that the power series (43) has

[4] It may seem that around $v=0$ the argument breaks down since here s is not small, but this is only an apparent difficulty. In fact, s/T and Tx are both independent of v and these are the relevant quantities.

[5] This is a Laplace transform that can be found, e.g., in H. Bateman, *Tables of Integral Transforms* (McGraw-Hill Book Co., New York, 1954), Vol. I, p. 150, Eq. (3).

the radius of convergence

$$\lim_{n\to\infty} [(n+1)^2(2n+2)^{-1}(2n+1)^{-1}][1+(n+1)^2\kappa^2] = \lim_{n\to\infty} [\tfrac{1}{4}(n+1)^2\kappa^2] = \infty\,. \quad (45)$$

The power series (43) thus diverges for $\kappa=0$ (when $I>0.5$), but converges as soon as $\kappa\neq 0$. For small values of κ the convergence, naturally, is slow, but for large $\kappa\gg 1$ we find

$$N(v)=\bar{N}\sum_{\alpha=0}^{\infty}(-1)^\alpha(2\alpha)!(\alpha!)^{-4}\left(\frac{\wp E}{2Kv}\right)^{2\alpha}, \quad (46)$$

which is a rapidly converging series, partly because of the extra factors $(\alpha!)^2$ in the denominators, and partly because the expansion parameter now is $\wp E/2Kv$, which in the limit of large κ is much smaller than the expansion parameter $\wp E/2\gamma$, which is relevant in the region of small κ. For very large v, only the first term of (46) will contribute, and N will become independent of v as it should.

The series (43) has been summed numerically near $v=0$ and for $I=3.6$, and the results are shown in Fig. 1. The existence of the bump in the bottom of the hole in the population inversion density is verified, and if Fig. 1 is compared to Fig. 12 of Ref. 1, we find that the series expression (43) leads to a shape of the hole very similar to the one obtained from the continued-fraction method. The series expansion (43) gives a somewhat better convergence than the power-series expansion method used in Ref. 1, which was based on the Fourier expansion. The lower part of Fig. 1 shows the number of terms necessary for convergence of the series (43), and we can see that below $\kappa=1$ a rapid increase occurs when κ decreases, and that at $\kappa=0$ the series does no longer converge. A velocity average will include a region with small values of v in which the usefulness of the power-series expansion is questionable unless $I\ll 1$, and then the third-order theory of Lamb[2] suffices.

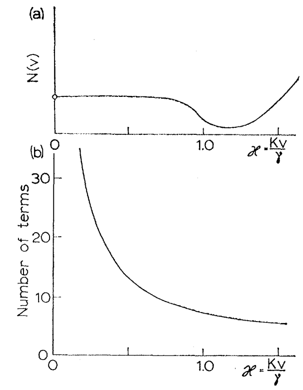

FIG. 1. (a) Shape of the function $N(v)$ near the value $v=0$ for $\omega=\nu$ and $I=3.6$. The position of the horizontal axis does not indicate the zero level of the vertical axis. The point at $\kappa=0$ is obtained from Eq. (44). (b) The number of terms needed in the power series to achieve convergence. The rapid increase below $\kappa=1.0$ is noted. At $\kappa=0$, the series diverges.

VI. DISCUSSION

The solution of the semiclassical laser equations given by Eqs. (19) and (20) is restricted mainly because it treats only the case $\omega=\nu$. It is, however, possible to discuss several aspects of the solution. The parameter $\wp E/\hbar Kv$ which occurs inside the trigonometric functions may be used for a series expansion, but may easily cause divergences for low values of v unless the factors

$$\begin{gathered}\cos(Kv\eta)-1=-\tfrac{1}{2}(Kv\eta)^2+\cdots,\\ \sin Kv\eta=Kv\eta+\cdots\end{gathered} \quad (47)$$

are treated properly. In the case of $N(v)$ which was treated in Sec. V, we saw that the v dependence appears in a rather complicated way in the final answer determining the convergence of the whole series. Since the trigonometric functions have no asymptotic forms for large arguments, the rather involved discussion of Sec. IV had to be used in order to obtain an expansion in E^{-1}. The proof presented does not claim any high standard of mathematical rigor, but it seems adequate. No asymptotic limit of the continued fraction analysis of Ref. 1 has been found.

Recently, Greenstein[6] has presented a laser theory which essentially is equivalent to the lowest-order continued-fraction approximation (called REA in Ref. 1). This is a very good approximation, especially when one calculates only quantities integrated over the velocity, but it is bound to give the wrong asymptotic behavior for large values of I, since this behavior is reached just for those values of I where the lowest-order approximation fails.

Even if the solution presented here is given in a closed analytic form, it contains a threefold integral which has not been evaluated. To obtain numerical results from these integrations is a rather cumbersome procedure. The corresponding expression in the method of Ref. 1 contains only one integration (the velocity average). The integrand is then not given in a closed form but as a continued fraction that is rapidly convergent and suitable for numerical work. The general conclusion is, therefore, that if numerical answers are wanted, the results of Ref. 1 are superior to the expressions of this paper; but if one wants to answer questions of a more general nature about the solutions of the laser equations, like the ones in Secs. IV and V, the exact analytic results may prove useful.

Note added in proof. The special case solved in this paper has also been solved exactly by B. J. Feldman and M. S. Feld, MIT (to be published), by the Fourier expansion method of Ref. 1. They find that the difference equations for the Fourier coefficients of this special case can be written as a recursion relation between Bessel's functions. They are thus able to connect the present work with that of Ref. 1. Their work also shows that the Fourier series are convergent in contradistinction to the asymptotic convergence claimed in Ref. 1.

[6] H. Greenstein, Phys. Rev. **175**, 438 (1968).

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APPENDIX

We want to consider the contributions to the integral

$$I_3=\int_0^{n\pi} dy\,\sin y\int_0^{\infty} dx\,e^{-sx}\times\exp\{iT[\cos(y-x)-\cos y]\} \quad \text{(A1)}$$

from the points of stationary phase

$$x=y+n\pi \quad (x>0), \quad \text{(A2)}$$

when $T\rightarrow\infty$. For all $y\neq 0$, the points (A2) do not coincide with the point $x=0$. It is enough to consider what happens at one of the points (A2), and for simplicity we choose the one with $n=0$. Then

$$\cos(y-x)=1-\tfrac{1}{2}(x-y)^2+\cdots, \quad \text{(A3)}$$

which gives for large values[3] of T

$$\begin{aligned}\exp[iT(1-\cos y)]&\int_0^{\infty} e^{-sx}\exp[-\tfrac{1}{2}iT(x-y)^2]dx\\&=\exp[iT(1-\cos y)]e^{-ys}\left(\frac{2}{iT}\right)^{1/2}\times\int_{-\infty}^{+\infty}\exp\left[-s\left(\frac{2}{iT}\right)^{1/2}t-t^2\right]dt\\&\sim\exp[iT(1-\cos y)]e^{-ys}\left(\frac{2}{iT}\right)^{1/2}\sqrt{\pi}. \quad \text{(A4)}\end{aligned}$$

Next we take the integral

$$I_3\sim\int_0^{n\pi} dy\,\sin y\left(\frac{2\pi}{iT}\right)^{1/2} e^{-sy}\exp[iT(1-\cos y)]. \quad \text{(A5)}$$

The integration over y has points of stationary phase when

$$(d/dt)\cos y=-\sin y=0, \quad \text{(A6)}$$

i.e., when

$$y=k\pi \quad (k=0,\,1,\,2,\cdots). \quad \text{(A7)}$$

But then the factor $\sin y$ gives zero for the integral, and a more detailed discussion is necessary. It is enough to consider the integral

$$I_4=\int_0^{\pi} dy\,\sin y\,e^{-sy}\exp(-iT\cos y), \quad \text{(A8)}$$

because the following periods only repeat the same behavior suppressed by the factor e^{-sy}.

We write

$$I_4=\exp(-s\bar{y})\int_0^{\pi} dy\,\sin y\,e^{-iT\cos y}=2\exp(-s\bar{y})\frac{\sin T}{T}\propto T^{-1}, \quad \text{(A9)}$$

where $\bar{y}$ is some value of y in the interval $(0,\pi)$. The contributions from the different intervals of length π in the integral (A5) all look like (A9) but have different signs and different values for $\bar{y}$. Thus it has been shown that for the point (A2) with $n=0$, we have

$$I_3\propto T^{-3/2}. \quad \text{(A10)}$$

The contribution from every point is of the same form, and the only part of the integral that behaves differently is the exponential, and this adds factors of the type $e^{-n\pi s}$. These never cause any trouble because n is bounded below

$$n>y/\pi. \quad \text{(A11)}$$